Homework 5
Real Analysis

Joshua Ruiter

March 23, 2018

Proposition 0.1 (Exercise 5a). *Let F be a closed subset of $\mathbb{R}$ such that $m(\mathbb{R} \setminus F)$ is finite. Define*

$$\delta(x) = d(x, F) = \inf\{|x - a| : a \in F\}$$

Then $\delta : \mathbb{R} \to \mathbb{R}$ is continuous.

I'm not sure how to prove the above statement at the moment.

Proposition 0.2 (Exercise 5b). *Let F be a closed subset of $\mathbb{R}$ with $m(\mathbb{R} \setminus F) < \infty$. Define*

$$\delta(x) = d(x, F) = \inf\{|x - a| : a \in F\}$$
$$f_x(y) = \frac{\delta(y)}{|x - y|^2}$$
$$I(x) = \int_{\mathbb{R}} f_x(y) dy$$

Then $I(x) = \infty$ for $x \notin F$.

Proof. For $n \in \mathbb{N}$ and $x \in \mathbb{R} \setminus F$, set $g_n^x(y) = \min(f_x(y), n)$ and let $E_n^x = \{y \in \mathbb{R} \setminus F : f_x(y) > n\}$. We have $g_n^x \nearrow f_x$, so by the Monotone Convergence Theorem,

$$I(x) = \int f_x(y) dy = \lim_{n \to \infty} \int g_n^x = \lim_{n \to \infty} \left(\int_{\mathbb{R} \setminus E_n^x} g_n^x + \int_{E_n^x} g_n^x \right) = \lim_{n \to \infty} \left(\int_{\mathbb{R} \setminus E_n^x} f_x + \int_{E_n^x} n \right)$$

Since F is closed, $\mathbb{R} \setminus F$ is open, so there is an open ball $B(x, r)$ contained in $\mathbb{R} \setminus F$. Since δ is continuous, f_x is continuous on $B(x, r) \setminus \{x\}$. We know that as y approaches x, $f_x(y) \to \infty$, so there exists $M \in \mathbb{R}$ such that $f_x(B(x, r)) = (M, \infty)$. Then for $n \in \mathbb{N}$, we can choose $a_n, b_n \in B(x, r) \setminus \{x\}$ with $a_n < x < b_n$ and $f_x(a_n), f_x(b_n) > n$. We then have $m(E_n^x) \geq m(a_n, b_n) = b_n - a_n$ so

$$\int_{E_n^x} n = nm(E_n) \geq n(b_n - a_n) = f(a_n)(a_n - b_n)$$
$$\geq f(a_n)(x - a_n) = \frac{\delta(a_n)}{(x - a_n)^2}(x - a_n) = \frac{\delta(a_n)}{x - a_n}$$

As $n \to \infty$, $a_n \to x$, so

$$\lim_{n\to\infty} \int_{E_n^x} n = \infty$$

Thus

$$I(x) \geq \lim_{n\to\infty} \int_{E_n^x} n = \infty$$

so $I(x) = \infty$. □

Proposition 0.3 (Exercise 5c)**.** *Let F be a closed subset of $\mathbb{R}$ such that $m(\mathbb{R} \setminus F)$ is finite. Define*

$$\delta(x) = d(x, F) = \inf\{|x - a| : a \in F\}$$
$$I(x) = \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} dy$$

Then $I(x) < \infty$ for almost all $x \in F$.

Proof. We will show that

$$\int_F I(x)dx < \infty$$

From this it will follow that $I(x) < \infty$ for almost all $x \in F$. First, note that $\delta(y) = 0$ for $y \in F$, so

$$I(x) = \int_{\mathbb{R}\setminus F} \frac{\delta(y)}{|x-y|^2} dy$$

Then we can compute

$$\int_F I(x)dx = \int_F \int_{\mathbb{R}\setminus F} \frac{\delta(y)}{|x-y|^2}\, dy\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} \chi_{F\times(\mathbb{R}\setminus F)}\, dy\, dx$$

Note that $\frac{\delta(y)}{|x-y|^2}\chi_{F\times(\mathbb{R}\setminus F)}$ is measurable since δ is continuous (and hence measurable) and $\frac{1}{|x-y|^2}$ is continuous almost everywhere, so this is a product of measurable functions. They by Fubini's Theorem, we can interchange the order of integration to get

$$\begin{aligned}\int_F I(x)dx &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\delta(y)}{|x-y|^2} \chi_{(\mathbb{R}\setminus F)\times F}\, dx\, dy \\ &= \int_{\mathbb{R}\setminus F} \int_F \frac{\delta(y)}{|x-y|^2}\, dx\, dy \\ &= \int_{\mathbb{R}\setminus F} \delta(y) \int_F \frac{1}{|x-y|^2} dx\, dy\end{aligned}$$

Now, becase $F \subset \{(x,y) : |x-y| \geq \delta(y)\}$, we have

$$\int_F \frac{1}{|x-y|^2} dx \leq \int_{-\infty}^{\delta(y)} \frac{1}{x^2} dx + \int_{\delta(y)}^{\infty} \frac{1}{x^2} dx = 2 \int_{\delta(y)}^{\infty} \frac{1}{x^2} dx$$

This is a Riemann-integrable function, so we can compute the integral using standard techniques to get

$$\int_{\delta(y)}^{\infty} \frac{1}{x^2} dx = \frac{1}{\delta(y)}$$

Putting this together, we have

$$\int_F I(x) dx \leq \int_{\mathbb{R} \setminus F} \delta(y) \frac{2}{\delta(y)} \, dy = 2m(\mathbb{R} \setminus F)$$

As $m(\mathbb{R} \setminus F) < \infty$ by hypothesis, we have what we set out to prove:

$$\int_F I(x) dx < \infty$$

Thus $I(x) < \infty$ for almost all $x \in F$. □

Proposition 0.4 (Exercise 6a)**.** *There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$ but* $\limsup_{x \to \infty} f(x) = \infty$.

Proof. First, let $g : \mathbb{R} \to [0, \infty)$ be

$$g(x) = \begin{cases} k & x \in [k, k + 1/k^3) \quad \text{for } k \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

We can approximate g by the sequence of simple functions

$$g_n(x) = \begin{cases} k & x \in [k, k + 1/k^3) \quad \text{for } k \in \mathbb{N}, k \leq n \\ 0 & \text{else} \end{cases}$$

These clearly converge to g everywhere. The integral of g_n is

$$\int g_n(x) = \sum_{k=1}^{n} \frac{k}{k^3} = \sum_{k=1}^{n} \frac{1}{k^2}$$

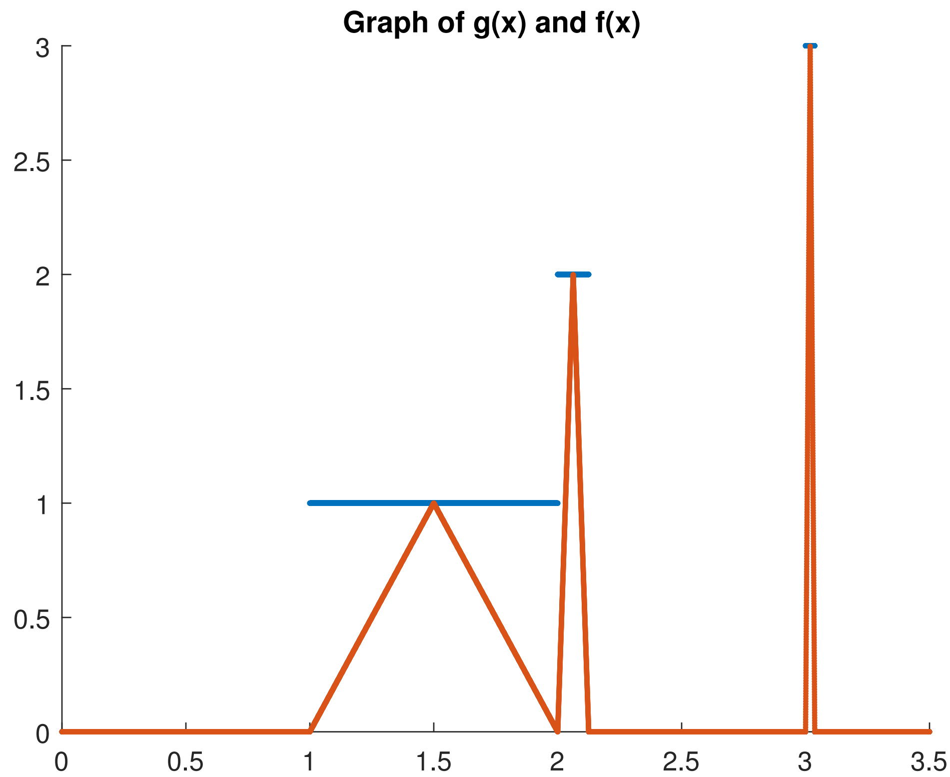
And since g_n are increasing to g, the integral of g is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

which is notably finite. We will make some modifications to find a continuous function analogous to g. Define $f : \mathbb{R} \to [0, \infty)$ by

$$f(x) = \begin{cases} 2n^4 x - 2n^5 & x \in [n, n + 1/(2n^3)) \\ -2n^4 x + 2n^5 + 2n & x \in [n + 1/(2n^3), n + 1/n^3) \\ 0 & \text{else} \end{cases}$$

Here is a plot of the graphs of f and g on the interval $[0, 3.5]$. Orange is f, blue is g. In places where you don't see blue, g is zero (the plot of f is on layered on top of the plot of g).



As defined, f is piecewise linear and the pieces meet at the breaks, so f is continuous. By monotonicity,

$$\int f(x) \leq \int g(x)$$

so f is integrable. However, we note that clearly

$$\limsup_{x\to\infty} f(x) = \lim_{y\to\infty} \sup\{f(x) : x \geq y\} = \infty$$

Thus f is a non-negative function which is integrable on $\mathbb{R}$ but has $\limsup_{x\to\infty} f(x) = \infty$. If we really want a function that is positive everywhere and still enjoys this property, we can use

$$h(x) = f(x) + \frac{1}{1+x^2}$$

as $1/(1+x^2)$ is continuous and gives a finite integral over $\mathbb{R}$. □

Lemma 0.5 (for Exercise 6b)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, and we set*

$$f^+ = \max(f, 0)$$
$$f^- = \max(-f, 0)$$

then f^+ and f^- are uniformly continuous.

Proof. Let $\epsilon > 0$. By uniform continuity of f, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

We claim that

$$|f^+(x) - f^+(y)| \leq |f(x) - f(y)|$$

Without loss of generality, assume $f(y) \leq f(x)$. If both $f(x), f(y)$ are positive, then we have equality, so we have the desired inequality. If both are less than or equal to zero, then $f^+(x) = f^+(y) = 0$ so we have the inequality trivially. Finally, if $f(y) \leq 0$ and $f(x) > 0$, then $f^+(x) = f(x)$ and $f^+(y) = 0 \geq f(y)$, so the inequality holds. Thus

$$|x - y| < \delta \implies |f^+(x) - f^+(y)| < \epsilon$$

so f^+ is uniformly continuous. An analogous argument holds for f^-. □

Proposition 0.6 (Exercise 6b). *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous and integrable. Then*

$$\lim_{|x| \to \infty} f(x) = 0$$

Proof. First we consider the case where f is non-negative. Let f be uniformly continuous and integrable, and suppose that $\lim_{|x|\to\infty} f(x) \neq 0$. Then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there exists a_n such that $a_n > n$ and $f(a_n) > 2\epsilon$. By skipping terms if necessary, we can choose a_n so that $a_n + 1 < a_{n+1}$ (since $\lim a_n = \infty$).

Since f is uniformly continuous and $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in \mathbb{R}$,

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

We can assume $\delta \leq 1/2$ (if not, we simply set $\delta = 1/2$). Consider the union

$$A = \bigcup_{n=1}^{\infty} B(a_n, \delta)$$

Note that this union is disjoint, as $d(a_n, a_k) \geq 1$ for all $n \neq k$ and $\delta \leq 1/2$. Set $g = \epsilon\chi_A$. We claim that $g \leq f$. On $\mathbb{R} \setminus A$ this is obvious as f is nonnegative and $g = 0$. On A, we have $g(x) = \epsilon$. For $x \in A$, we have

$$|x - a_n| < \delta \implies |f(x) - f(a_n)| < \epsilon \implies f(x) > \epsilon$$

since $f(a_n) > 2\epsilon$. Thus we have established that $g \leq f$. Now consider the integral of g.

$$\int g(x)dx = \int \epsilon\chi_A dx = \epsilon m(A) = \epsilon \sum_{n=1}^{\infty} m(B(a_n, \delta)) = \epsilon \sum_{n=1}^{\infty} 2\delta$$

Since $\epsilon, \delta > 0$, this sum is infinite. By monotonicity,

$$\int f \geq \int g = \infty$$

This contradicts the hypothesis that f is integrable, so we reject our assumption that $\lim_{|x|\to\infty} f(x) \neq 0$ and conclude that the limit is zero.

Now we return to the general case, where f is any integrable and uniformly continuous function, that may not be non-negative. Then we can write f as $f = f^+ - f^-$ where f^+, f^- are integrable, and by the above lemma, they are also uniformly continuous. Since both are non-negative, the result just shown applies, so each has the desired limit. Then by limit laws

$$\lim_{|x|\to\infty} f = \lim_{|x|\to\infty} f^+ - \lim_{|x|\to\infty} f^- = 0 - 0 = 0$$

□

Proposition 0.7 (Exercise 8)**.** *Let f be integrable on $\mathbb{R}$ and define $F : \mathbb{R} \to \mathbb{R}$ by*

$$F(x) = \int_{-\infty}^{\infty} f(t)dt$$

Then F is uniformly continuous.

Proof. We need to show that for $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |F(x) - F(y)| < \epsilon$$

Let $\epsilon > 0$. Without loss of generality, assume that $x < y$. Note that

$$\int_{-\infty}^{y} f(t)dt = \int_{-\infty}^{x} f(t)dt + \int_{x}^{y} f(t)dt$$

So then

$$\begin{aligned}
|F(x) - F(y)| &= \left| \int_{\infty}^{x} f(t)dt - \int_{-\infty}^{y} f(t)dt \right| \\
&= \left| \int_{-\infty}^{x} f(t)dt - \int_{-\infty}^{x} f(t)dt - \int_{x}^{y} f(t)dt \right| \\
&= \left| \int_{x}^{y} f(t)dt \right| \\
&\leq \int_{x}^{y} |f(t)|dt
\end{aligned}$$

By Proposition 1.12(ii), there exists $\delta > 0$ such that

$$\int_E |f| < \epsilon$$

whenever $m(E) < \delta$. Then if $|x - y| < \delta$, $m((x, y)) < \delta$ so

$$\int_{x}^{y} |f(t)|dt < \epsilon$$

so combining our inequalities, we reach the desired inequality.

$$|F(x) - F(y)| \leq \int_{x}^{y} |f(t)|dt < \epsilon$$

whenever $|x - y| < \delta$. Thus F is uniformly continuous. □

Proposition 0.8 (Exercise 9)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be integrable and non-negative. For $\alpha > 0$, define*

$$E_\alpha = \{x : f(x) > \alpha\}$$

Then

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f$$

Proof. Define $g_\alpha : \mathbb{R} \to \mathbb{R}$ by $g_\alpha = \alpha \chi_{E_\alpha}$. Then $0 \leq g_\alpha \leq f$, and $\int g_\alpha = \alpha m(E_\alpha)$, so by monotonicity,

$$\int g_\alpha \leq \int f \implies \alpha m(E_\alpha) \leq \int f \implies m(E_\alpha) \leq \frac{1}{\alpha} \int f$$

□

Lemma 0.9 (for Exercise 10)**.** *Let $a_{nk} \in \mathbb{R}^d$ be a sequence over a countable indexing set $\{(n,k) \in N \times K\}$, such that $a_{nk} \geq 0$ for all n, k. Then*

$$\sum_{n \in N} \sum_{k \in K} a_{nk} = \sum_{k \in K} \sum_{n \in N} a_{nk}$$

Proof. Without loss of generality, suppose $N = K = \mathbb{N}$. Define $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} a_{nk} & n \leq x < n+1 \text{ and } k \leq y < k+1 \\ 0 & \text{otherwise} \end{cases}$$

Then we have

$$\int_{\mathbb{R}} f(x,k)\, dx = \sum_{n \in N} a_{nk}$$
$$\int_{\mathbb{R}} f(n,y)\, dy = \sum_{k \in K} a_{nk}$$

So

$$\sum_{k \in K} \sum_{n \in N} = \int f(x,y)\, dx\, dy$$
$$\sum_{n \in N} \sum_{k \in K} = \int f(x,y)\, dy\, dx$$

By Fubini's Theorem, these two integrals are equal. □

Lemma 0.10 (for Exercise 10)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be measurable and $\{E_k\}_{k=1}^\infty$ be a disjoint sequence of sets. Let $E = \bigcup_{k=1}^\infty E_k$. Then*

$$\int_E f = \sum_{k=1}^\infty \int_{E_k} f$$

Proof. Let $g = \chi_E f$ and $F_n = \bigcup_{k=1}^n E_k$ and $g_n = \chi_{F_n} f$. Then $g_n \nearrow g$ so by the Monotone Convergence Theorem we get

$$\lim_{n\to\infty} \int_{F_n} f = \lim_{n\to\infty} \int \chi_{F_n} f = \lim_{n\to\infty} \int g_n = \int g = \int_E f$$

By additivity,

$$\int_{F_n} f = \sum_{k=1}^{n} \int_{E_k} f$$

Thus

$$\int_E f = \lim_{n\to\infty} \sum_{k=1}^{n} \int_{E_k} f = \sum_{k=1}^{\infty} \int_{E_k} f$$

□

Proposition 0.11 (Exercise 10)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be non-negative and measurable, and finite almost everywhere. Let $E_{2^k} = \{x : f(x) > 2^k\}$ and $F_k = \{x : 2^k < f(x) \le 2^{k+1}\}$. The following are equivalent:*

1. *f is integrable*
2. *$\sum_{k\in\mathbb{Z}} 2^k m(F_k) < \infty$*
3. *$\sum_{k\in\mathbb{Z}} 2^k m(E_{2^k}) < \infty$*

Proof. First we show (1) $\implies$ (2). Let f be integrable, that is, suppose $\int f < \infty$. We know that

$$2^k m(F_k) \le \int_{F_k} f$$

since $f(x) \ge 2^k$ on F_k. Since the collection $\{F_k\}$ is pairwise disjoint, we can use our lemma to get

$$\sum_{k\in\mathbb{Z}} 2^k m(F_k) \le \sum_{k\in\mathbb{Z}} \int_{F_k} f = \int f < \infty$$

Thus (2) holds. Now we show that (2) $\implies$ (1). By our lemma,

$$\int f = \sum_{k\in\mathbb{Z}} \int_{F_k} f$$

We know that $f(x) \le 2^{k+1}$ for $x \in F_k$,so

$$\int_{F_k} f \le 2^{k+1} m(F_k)$$

thus

$$\int f \leq \sum_{k\in\mathbb{Z}} 2^{k+1} m(F_k) = 2\sum_{k\in\mathbb{Z}} 2^k m(F_k) < \infty$$

thus f is integrable, so (1) holds. Now we show that (3) $\implies$ (2). Suppose that (3) holds. Note that $F_k \subset E_{2^k}$ so $m(F_k) \leq m(E_{2^k})$, thus

$$2^k m(F_k) \leq 2^k m(E_{2^k}) \implies \sum_{k\in\mathbb{Z}} 2^k m(F_k) \leq \sum_{k\in\mathbb{Z}} 2^k m(E_{2^k}) < \infty$$

thus (2) holds. Finally, we show that (2) $\implies$ (3). Suppose that (2) holds. Note that $E_{2^k} = \bigcup_{n\geq k} F_n$ (this union is disjoint). Then

$$\sum_{k\in\mathbb{Z}} 2^k m(E_k) = \sum_{k\in\mathbb{Z}} 2^k \left(\sum_{n\geq k} m(F_n)\right) = \sum_{k\in\mathbb{Z}}\sum_{n\geq k} 2^k m(F_n)$$

We can interchange these limits by our lemma, so

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^k m(E_{2^k}) &= \sum_{n\in\mathbb{Z}}\sum_{k\leq n} 2^k m(F_n) = \sum_{n\in\mathbb{Z}} m(F_n)\sum_{k\leq n} 2^k = \sum_{n\in\mathbb{Z}} m(F_n)\sum_{j=0}^{\infty} 2^{n-j} \\ &= \sum_{n\in\mathbb{Z}} m(F_n) 2^n \sum_{j=0}^{\infty} 2^{-j} = \sum_{n\in\mathbb{Z}} 2^{n+1} m(F_n) = 2\sum_{n\in\mathbb{Z}} 2^n m(F_n) < \infty\end{aligned}$$

Thus (3) holds. We showed (1) $\iff$ (2) and (2) $\iff$ (3), so they are all equivalent. □

Proposition 0.12 (Exercise 10)**.** *Define*

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} |x|^{-b} & \text{if } |x| > 1 \\ 0 & \text{otherwise} \end{cases}$$

Then f is integrable on $\mathbb{R}^d$ if and only if $a < d$; also g is integrable on $\mathbb{R}^d$ if and only if $b > d$.

Proof. First consider f. Then

$$F_k = \{x : 2^k < f(x) \leq 2^{k+1}\} = \{x \in \overline{B}(0,1) : 2^k < |x|^{-a} \leq 2^{k+1}\}$$

For $k < 0$, $F_k = \emptyset$. For $k \geq 0$, F_k is a spherical annulus between balls of radius $2^{-k/a}$ and $2^{-(k+1)/a}$. So we can compute $m(F_k)$ as

$$\begin{aligned} m(F_k) &= m(B(0, 2^{-k/a})) - m(B(0, 2^{-(k+1)/a})) \\ &= v_d (2^{-k/a})^d - v_d (2^{-(k+1)/a})^d \\ &= v_d 2^{-kd/a}(1 - 2^{-d/a}) \end{aligned}$$

where v_d is the constant we found in Exercise 6 of Chapter 1. So then

$$\sum_{k\in\mathbb{Z}}^{\infty} 2^k m(F_k) = \sum_{k=0}^{\infty} 2^k v_d 2^{-kd/a}(1-2^{-d/a}) = v_d(1-2^{d/a}) \sum_{k=0}^{\infty} (2^k)^{(1-d/a)}$$

If $a < d$, then $1 - d/a < 0$ so the sum converges. Conversely, if the sum converges, we must have $1 - d/a < 0$, so $a < d$. Hence f is integrable if and only if $a < d$.

Now we consider g. For g,

$$E_{2^k} = \{x : g(x) > 2^k\} = \{x \in \mathbb{R} \setminus \overline{B}(0,1) : |x|^{-b} > 2^k\}$$

For $k \geq 0$, $E_{2^k} = \emptyset$. For $k < 0$, E_{2^k} is again a difference of balls centered at the origin, so

$$\begin{aligned} m(E_{2^k}) &= m(B(0, 2^{-k/b}) - m(B(0,1)) \\ &= v_d 2^{-kd/b} - v_d \\ &= v_d(2^{-kd/b} - 1) \end{aligned}$$

So we compute

$$\begin{aligned} \sum_{k\in\mathbb{Z}} 2^k m(E_{2^k}) &= \sum_{k<0} 2^k v_d (2^{-kd/b} - 1) \\ &= v_d \sum_{k<0} 2^{k(1-d/b)} - 2^k \\ &= v_d \sum_{k<0} 2^{k(1-d/b)} - v_d \sum_{k<0} 2^k \end{aligned}$$

The right sum converges to 1, and the left sum converges if and only if $d < b$. Thus g is integrable if and only if $b > d$. □

Proposition 0.13 (Exercise 11)**.** *Suppose that f is integrable on $\mathbb{R}^d$, real-valued, and $\int_E f(x)dx \geq 0$ for every measurable $E \subset \mathbb{R}^d$. Then $f(x) \geq 0$ almost everywhere.*

Proof. For $E \subset \mathbb{R}^d$ and $\epsilon > 0$, define

$$E_\epsilon = \{x \in E : f(x) \leq -\epsilon < 0\}$$

Note that since f is measurable, E_ϵ is measurable. Observe that

$$\bigcup_{n=1}^{\infty} E_{1/n} = \{x : f(x) < 0\}$$

By hypothesis, for every n we have

$$0 \leq \int_{E_{1/n}} f(x)dx \leq -\frac{m(E_{1/n})}{n} \implies m(E_{1/n}) = 0$$

Suppose the proposition is false. Then there exists a measurable $E \subset \mathbb{R}^d$ with $m(E) > 0$ and $f(x) < 0$ on E. Then

$$\bigcup_{n=1}^{\infty} E_{1/n} = E \implies \sum_{n=1}^{\infty} m(E_{1/n}) = m(E)$$

but the infinite sum has all terms zero, so $m(E) = 0$, which is a contradiction. □

Proposition 0.14 (Exercise 11)**.** *Let f be integrable on $\mathbb{R}^d$, real-valued, and $\int_E f = 0$ for every measurable set $E \subset \mathbb{R}^d$. Then $f(x) = 0$ almost everywhere.*

Proof. As shown above, $f(x) \geq 0$ almost everywhere. Additionally, $\int_E -f = 0$ for every measurable E, so by the previous result $-f(x) \geq 0 \implies f(x) \leq 0$ almost everywhere. Thus

$$f(x) \leq 0 \leq f(x)$$

almost everywhere, so $f(x) = 0$ almost everywhere. □